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## Wavelets in mathematical physics: $q$ -oscillators

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### Abstract

We construct representations of a  $q$ -oscillator algebra by operators on Fock space on positive matrices. They emerge from a multiresolution scaling construction used in wavelet analysis. The representations of the Cuntz algebra arising from this multiresolution analysis are contained as a special case in the Fock space construction.

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In this paper we establish a connection between multiresolution wavelet analysis on one hand and representation theory for operator on Hilbert spaces depending on a real parameter on the other. These operators arise from a multiresolution wavelet analysis based on Bessel functions. We wish to develop a framework for the study of creation operators on Hilbert space, satisfying simple identities, and allowing a Hopf algebra structure. Examples will include oscillator algebras coming from physical models.

In the first section of the paper, we review the background and the motivation for the study of the  $q$ -relations, both as it relates to problems in mathematics and in physics. On the mathematical side, the problems concern wavelet analysis and transform theory, especially the Mellin transform, and on the physics side, they relate to the quon gas of statistical mechanics. For the construction of the representations, we then turn to the twisted Fock space and the  $q$ -oscillator algebra. Our approach is motivated by wavelet analysis, and it uses a certain loop group. Our main result is theorem 6.

### 1. Introduction

Some of the results from the papers [2, 3, 6] are based on an operator-theoretic approach to wavelet theory involving representing wavelets in terms of operators in an infinite-dimensional Hilbert space.

In this paper we introduce an analogous operator approach in a study of a generalized biorthogonal wavelet, leading to the construction of oscillator algebras, and more generally

Hopf algebras. In [7] a related class of representations of the Cuntz algebra  $\mathcal{O}_N$  have been found depending on a parameter  $q$ .

We shall distinguish between two aspects of the study of the  $q$ -relations: (i) the  $C^*$ -algebra on these relations, and (ii) finding the representations of them. While the paper [8] by Jorgensen *et al* covered (i), we shall concentrate here on (ii). This is a difficult problem: as noted in [8], the  $C^*$ -algebra  $\mathfrak{A}(q)$  on the  $q$ -relations is infinite, and its equivalence classes of irreducible representations do not admit a Borel cross section for its classification parameters. In [8], the authors showed that there is a stability interval  $J$  in the  $q$ -variable such that all the  $C^*$ -algebra  $\mathfrak{A}(q)$  for  $q \in J$  are isomorphic to the Cuntz algebra, see [1], and they estimated the size of  $J$ .

The motivation for the problem (ii) comes from two sources, (a) from analysis (wavelets, special functions and combinatorics), and (b) from physics (quantum optics, statistical mechanics, quantum fields and anyons). We show in this paper how these problems may perhaps be understood better via the approach of  $q$ -deformations, and via the study of concrete mathematical settings where the representations arise naturally.

Other papers which cover representations include [1], [10] and [11]. The physics of the  $q$ -relations is outlined in [1, 4, 12]. In particular [12] relates the  $q$ -representations to the Gibbs paradox.

The  $C^*$ -algebra  $\mathcal{O}_N$ , called the Cuntz  $C^*$ -algebra on  $N$  generators, is universal on the relations

$$s_i^* s_j = \delta_{ij} \mathbf{1} \quad \sum_{i=1}^N s_i s_i^* = \mathbf{1} \quad (1)$$

where  $\mathbf{1}$  denotes the unit element in  $\mathcal{O}_N$ . If  $m_1, \dots, m_N$  are given functions on  $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ , then the operator system

$$(S_i f)(z) = m_i(z) f(z^N) \quad f \in L^2(\mathbf{T}) \quad z \in \mathbf{T} \quad i = 1, \dots, N$$

satisfies the Cuntz relations (1) if and only if the functions are frequency subband filters for the wavelet multiresolution construction [9]. Then  $m_1$  is called the low-pass filter, and the others filters of the higher frequency bands.

The conditions on the functions may be stated in either one of the following two equivalent forms (a) or (b): Let  $\rho_N = e^{i2\pi/N}$ .

(a) The  $N \times N$  matrix

$$M(z) = \frac{1}{\sqrt{N}} (m_j(\rho_N^k z))_{j,k=1}^N$$

is unitary for all  $z \in \mathbf{T}$ , i.e.,

$$M(z)^* M(z) = I_N \quad z \in \mathbf{T}.$$

(b) The  $N \times N$  matrix  $A(z) = (A_{j,k}(z))$  given by

$$A_{j,k}(z) = \frac{1}{N} \sum_{\substack{w \in \mathbf{T} \\ w^N = z}} m_j(w) w^{-k}$$

is unitary for all  $z \in \mathbf{T}$ , i.e.,

$$A(z)^* A(z) = I_N \quad z \in \mathbf{T}.$$

To complete the picture of multiresolution analysis depending on a parameter we include here the case of the construction of a different multiresolution via the Mellin transform. We develop a finite-scale multiresolution analysis via Mellin transforms giving rise to wavelets depending on a parameter  $q$ ,  $0 < q < 1$ .

## 2. Mellin transforms

Let us first recall some facts about Mellin transforms. The Mellin transform of a function  $f(x)$  is given by

$$M(f(x); s) = \int_0^{\infty} x^{s-1} f(x) dx.$$

The inverse transform gives

$$f(x) = M^{-1}(F(s); x)$$

where we set  $M(f(x); s) = F(s)$ .

The behaviour of the Mellin transform under various coordinate transforms in  $x$  space is given by

$$\begin{aligned} M(f(ax); s) &= a^{-s} M(f(x); s) \\ M(f(x^a); s) &= a^{-1} M\left(f(x); \frac{s}{a}\right) \\ M(x^a f(x); s) &= M(f(x); s + a) \end{aligned}$$

as can be easily checked.

Let us give some preliminaries on standard multiresolution wavelet analysis of scale  $N$ . Define scaling by  $N$  on  $L^2(\mathbf{R})$  by

$$U(\xi)(x) = \xi(N^{-1}x)$$

and translation by 1 by

$$T(\xi)(x) = \xi(x - 1).$$

A scaling function is a function  $\phi \in L^2(\mathbf{R})$  such that if  $V_0$  is the closed linear span of all translated  $T^k\phi$ ,  $k \in \mathbf{Z}$ , then  $\phi$  has the following properties:

- (a)  $\{T^k\phi : k \in \mathbf{Z}\}$  is an orthonormal set in  $V_0$ ,
- (b)  $U\phi \in V_0$ ,
- (c)  $\bigwedge_{n \in \mathbf{Z}} U^n V_0 = \{0\}$ ,
- (d)  $\bigvee_{n \in \mathbf{Z}} U^n V_0 = L^2(\mathbf{R})$ ,

where the symbol  $\bigvee$  means ‘closed linear span’, and  $\bigwedge$  ‘intersection of subspaces’. We shall further use the notation  $U_j := U^j V_0$ ,  $j \in \mathbf{Z}$ . The system a.–d. is called a multiresolution analysis (MRA).

We then have the following result

**Theorem 1.** *Let  $\Phi \in L^2(\mathbf{R})$  be the Haar function,  $0 < q < 1$ ,  $q \in \mathbf{R}$ . There exists a sequence of subspaces  $U_j$ ,  $j \in \mathbf{Z}$ , such that*

$$\cdots U_j \subset U_{j-1} \subset \cdots \subset U_0 \subset U_{-1} \subset \cdots \quad (2)$$

*giving a multiresolution satisfying the above properties.*

**Proof.** To show that the system in (2) is a multi-resolution, we need to establish a scaling operator which moves in steps of one through the ladder of resolution spaces  $U_j$ ; we must identify the Hilbert space  $\mathcal{H}$ ; and finally we must show that  $\bigvee_j U_j = \mathcal{H}$ , and  $\bigwedge_j U_j = \{0\}$  where  $\bigvee$  and  $\bigwedge$  are the usual lattice operations in Hilbert space.

Let  $\Phi \in L^2(\mathbf{R})$  be the Haar function,  $0 < q < 1$ ,  $x \in \mathbf{R}$ . We want to prove that there exists a sequence of subspaces  $U_j$  giving rise to a multiresolution satisfying the above properties. Define

$$V_0 = \bigvee \{\Phi(q^k x) : k \in \mathbf{Z}, q < x < 1\}.$$

Let  $U\xi(x) = \frac{1}{\sqrt{N}}\xi(x^N)$  be the scaling operator and  $T\xi(x) = \xi(qx)$  playing the role of the ‘translation operator’. In this case  $T$  is a dilation. Then the set  $\{\Phi(q^k x) : k \in \mathbf{Z}\}$  is orthonormal in  $L^2(\mathbf{R}_+)$ ,  $q \in \mathbf{R}$ ,  $0 < q < 1$ , since the intervals are disjoint for  $h \neq k$ ,  $h, k \in \mathbf{Z}$ . It is easy to verify that  $U\Phi \in V_0$ ,

$$U\Phi(x) = \sum_k a_k \Phi(q^k x). \quad (3)$$

Then  $\bigwedge_{n \in \mathbf{Z}} U^n V_0 = \{1\}$ . Suppose  $\xi \in U^n V_0$ , with  $\xi(x) = U^n \Phi(x) = \sum_k a_k \Phi((q^k x)^{N^n}) \frac{1}{N^n}$ . Since any  $\xi \in L^2(\mathbf{R}_+)$  can be approximated by step functions and  $\mathbf{R}_+ = \bigcup_{k,n} [q^{(k+1)N^n}, q^{kN^n}]$ , it follows that

$$\bigvee_{n \in \mathbf{Z}} U^n V_0 = L^2(\mathbf{R}_+).$$

Hence the above properties are satisfied and the sequence  $V_j = U^j V_0$  of subspaces  $V_j$  associated to  $\Phi$  defines a multiresolution. Let  $\Phi \in L^2(\mathbf{R})$  satisfy (3). Define the operator  $W_\Phi: L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})$ , called the wave operator, by

$$(W_\Phi f)(x) = \sum_{k \in \mathbf{Z}} c_k \Phi(q^k x) = \sum_{k \in \mathbf{Z}} M(f(q^k x); s) \Phi(q^k x). \quad \square$$

**Lemma 2.** *The operator  $W_\Phi$  satisfies*

$$U W_\Phi = W_\Phi S \quad (4)$$

where  $S$  is given by

$$M(Sf)(s) = \sum_j a_j q^{-sj} M\left(f; \frac{s}{N}\right).$$

**Proof.** First, we have

$$\begin{aligned} (U W_\Phi f)(x) &= \frac{1}{\sqrt{N}} (W_\Phi f)(x^N) = \frac{1}{\sqrt{N}} \sum_{k \in \mathbf{Z}} M(f(q^{kN} x^N); s) \Phi(q^{kN} x^N) \\ &= \frac{1}{\sqrt{N}} \sum_{k \in \mathbf{Z}} q^{-ksN} M\left(f(x); \frac{s}{N}\right) \Phi(q^{kN} x^N) \\ &= \frac{1}{\sqrt{N}} \sum_{k \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} q^{-ksN - sj} a_j M\left(f(x); \frac{s}{N}\right) \Phi(q^{kN+j} x^N) \\ &= \frac{1}{\sqrt{N}} \sum_{k \in \mathbf{Z}} \left[ \sum_{l \in \mathbf{Z}} a_{l-Nk} q^{-ls} M(f(x); s) \right] \Phi(q^l x^N). \end{aligned}$$

On the other hand

$$\begin{aligned} W_\Phi(Sf)(z) &= \sum_{k \in \mathbf{Z}} M((Sf)(xq^k); s) \Phi(q^k x) \\ &= \sum_{k \in \mathbf{Z}} M((Sf)(xq^k); s) \Phi(q^{kN+j} x^N) \\ &= \sum_{k \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} q^{-kNs - sj} a_j M\left(f(x); \frac{s}{N}\right) \Phi(q^{kN+j} x^N) \\ &= \sum_{k \in \mathbf{Z}} \sum_{l \in \mathbf{Z}} a_{l-kN} q^{-ls} M\left(f(x); \frac{s}{N}\right) \Phi(q^l x^N). \end{aligned}$$

Thus

$$UW_\Phi = W_\Phi S$$

which is the identity (4), i.e.,  $W_\Phi$  intertwines the operators  $U$  and  $S$ .  $\square$

Using the fact that  $\{T^k \Phi : k \in \mathbf{Z}\}$  is an orthonormal set in  $L^2(\mathbf{R})$  we may define

$$F_\Phi: V_0 \longrightarrow L^2(\mathbf{R})$$

an isometry which sends

$$\xi \longmapsto m(s) = \sum_k a_k q^{-ks} \quad \xi(s) = \sum_k a_k \Phi(q^{-k}x).$$

Then

$$M(\xi(x); s) = m(s)M(\Phi(x); s)$$

where  $M(\xi(x); s)$  is the Mellin transform.

If  $\xi \in V_{-1} = U^{-1}V_0$  then  $U\xi \in V_0$  so we define

$$m_\xi = F_\Phi(U\xi) \in L^2(\mathbf{R})$$

and then

$$\begin{aligned} M(U\xi(x); s) &= M\left(\sum_k a_k \Phi((q^k x)^N); s\right) = \sum_k a_k M(\Phi((q^k x)^N); s) \\ &= \sum_k a_k M(\Phi(q^{kN} x^N); s) = \sum_k \frac{1}{N} a_k q^{-ksN} M\left(\Phi(x); \frac{s}{N}\right). \end{aligned}$$

Thus

$$NM(U\xi(x^N); s) = m_\xi(s)M(\Phi(x); s). \quad (5)$$

Observe that in this case we do not have unitarity of the matrix  $A$  generated by the filters  $m_\xi$ . Thus we do not have representations of the Cuntz algebra. This brings out the question of what are the natural operator relations associated to this MRA. One particular choice is given by the oscillator algebras.

For any MRA wavelet construction as found above we construct an appropriate oscillator algebra such that the filters make up the eigenfunctions system of the oscillator Hamiltonian, i.e., the energy operator.

In fact the non-unitarity of the matrix given by the MRA filters provides the eigenfunctions for the energy operator. The  $q$ -oscillator algebra is given by the operators  $a^-$ ,  $a^+$  and the number operator. By a standard argument it is possible to build up the following selfadjoint operators:

$$X = \frac{\sqrt{2}}{2}(a^+ + a^-) \quad \text{and} \quad P = \frac{\sqrt{2}}{2}(a^- - a^+)$$

which are the momentum and the position operators. Given the above MRA there exists a family of  $q$ -oscillators that can be represented via the filters.

Let  $H = m_0^2(s) + m_0^2(\sigma(s))$  be the energy operator. Let

$$a^-(s)a^+(s) = m_0(\sigma(s))^2$$

and

$$a^+(s)a^-(s) = m_0(s)^2.$$

Then it follows that the algebra is generated by  $a^+$ ,  $a^-$ ,  $N$  and the generators satisfy the following relations:

$$[a^-(s), a^+(s)] = f(N+1) - f(N) = f(N)[s]_{q^{-2}}$$

$$f(N)a^-(s) = a^-(s)f(N-1)$$

$$f(N)a^+(s) = a^+(s)f(N+1)$$

where  $f(N+1) - f(N) = \sum_k b_k q^{-2ks} [s]_{q^{-2}}$ .

Thus we have

$$[a^-(s), a^+(s)] = m_0(s)^2 [s]_{q^{-2}}$$

where  $[s]_q = \frac{1-q^s}{1-q}$ . Then using MRA we can construct a family of operators  $\{a_i^-(s), a_i^+(s)\}$  for  $i = 1, \dots, N$  satisfying

$$[a_i^-(s), a_i^+(s)] = m_i(s)^2 [s]_{q^{-2}}.$$

A representation of the operators  $a^-(s)$  and  $a^+(s)$  can be realized in a Fock space as follows:

$$a^-(s)|e_k\rangle = \frac{q}{1-q} \sum_k b_k q^{-ks} |e_{k-1}\rangle$$

and

$$a^+(s)|e_k\rangle = \frac{q}{1+q} \sum_k b_k q^{-(k+1)s} |e_{k+1}\rangle. \quad \square$$

### 3. Twisted Fock space

In this section we introduce a new Fock space construction, see [6], which may provide the appropriate framework for studying wavelet representations of certain  $q$ -oscillator algebras.

**Definition 3.** *The full Fock space over  $\mathbf{C}^N$  where  $N$  is a fixed positive integer with  $N \geq 2$ , is the orthogonal direct sum of Hilbert spaces given by*

$$K = \left( \sum_{k=-\infty}^{-1} \oplus (\mathbf{C}^N)^{\otimes -k} \right) \oplus \mathbf{C} = \dots \oplus (\mathbf{C}^N \otimes \mathbf{C}^N) \oplus (\mathbf{C}^N) \oplus \mathbf{C}.$$

*The term  $\mathbf{C}$  in the summand on the right designates the vacuum vector (in the formula we omit a special symbol  $\Omega$  for the vacuum vector). Let  $\{\xi_1, \dots, \xi_N\}$  be a fixed orthonormal basis for  $\mathbf{C}^N$ . Then  $K$  is an infinite-dimensional Hilbert space with orthonormal basis given by  $\{\xi_{i_1}, \dots, \xi_{i_k} : 1 \leq i_1, \dots, i_k \leq N, k \geq 1\} \cup \{\Omega\}$ .*

#### 3.1. Construction of twisted Fock space

Let  $K$  be a Hilbert space. We take the tensor product of the full Fock space with  $H$ , then we define a 'new' inner product  $\langle \cdot, \cdot \rangle_\Phi$  by using a completely positive map from the complex matrices into  $B(H)$  (i.e., a positive matrix with entries in  $B(H)$ ). By following [6] we construct the twisted Fock space as follows. Let  $\Phi: M_N \rightarrow B(H)$  be the completely positive map which we will define later. We define the  $N$ -variable pre-Fock space over  $K$  to be the vector space of finite sums

$$T_N(H) = \left\{ \sum_{|w| \leq k} w \otimes h_w : w \in \mathbf{F}_N^+, k \geq 1, h_w \in H \right\}$$

where  $\mathbf{F}_N^+$  is the unital free-semigroup on  $N$  non-commuting letters  $\{1, 2, \dots, N\}$  with unit  $e$ . We can think of the full Fock space as  $l^2(\mathbf{F}_N^+)$  where an orthonormal basis is given by the vectors  $\{\xi_w : w \in \mathbf{F}_N^+\}$  corresponding to the words  $w$ ,  $|w|$  is the word of length zero or empty word,  $\xi_w = \xi_{i_1} \otimes \dots \otimes \xi_{i_n}$ ,  $w = i_1 \dots i_k \in \mathbf{F}_N^+$ . Then a vector  $(i_1, \dots, i_k) \otimes h$  with  $w = i_1 \dots i_k \in \mathbf{F}_N^+$  corresponds to the vector  $\xi_{i_1} \otimes \dots \otimes \xi_{i_k} \otimes h$  in  $(\mathbf{C}^N)^{\otimes k} \otimes H$ .

Let  $\Phi$  be the completely positive map  $\Phi: M_N \longrightarrow B(H)$ . Define a form

$$\langle \cdot, \cdot \rangle_\Phi: T_N(H) \times T_N(H) \longrightarrow \mathbf{C}$$

as follows. For  $w, w' \in \mathbf{F}_N^+$ ,  $h, h' \in H$

- (i)  $\langle e \otimes h, e' \otimes h' \rangle_\Phi = \langle h | h' \rangle$ ;
- (ii) if  $|w| \neq |w'|$  then  $\langle w \otimes h, w' \otimes h' \rangle_\Phi = 0$ ;
- (iii) if  $w = i_1 \dots i_k$ ,  $w' = i'_1 \dots i'_k$ , then

$$\langle w \otimes h, w' \otimes h' \rangle_\Phi = \langle h | \Phi(e_{i_1 i'_1} \otimes \dots \otimes e_{i_k i'_k}) h' \rangle.$$

Extend  $\langle \cdot, \cdot \rangle_\Phi$  to  $T_N(H) \times T_N(H)$  as linear in the first variable and conjugate linear in the second one. From theorem 4.5 of [6] the form  $\langle \cdot, \cdot \rangle_\Phi$  is positive semi-definite on  $T_N(H)$ .

**Definition 4.** Let  $N_\Phi = \{x \in T_N(H) : \langle x | x \rangle_\Phi = 0\}$  be the kernel of the form  $\langle \cdot, \cdot \rangle_\Phi$ . Define the Fock space of  $\Phi$  over  $H$  to be the Hilbert space completion

$$F_N(K, \Phi) = \overline{T_N(H)/N_\Phi}^{\langle \cdot, \cdot \rangle_\Phi}.$$

The left creation operators  $T = (T_1, \dots, T_N)$  on  $\mathcal{F}_N(\mathcal{H}, \Phi)$  are linear transformations defined by

$$T_i(w \otimes h + N_\Phi) = (iw) \otimes h + N_\Phi.$$

These operators are well-defined and  $T_i(N_\Phi) \subset N_\Phi$ ,  $1 \leq i \leq N$ .

Let  $S_i$  be the  $i$ th creation operator on the first space defined as follows:

$$S_i x = S_i(\eta_{i_1} \otimes \dots \otimes \eta_{i_k} \otimes h) = \eta_i \otimes \eta_{i_1} \otimes \dots \otimes \eta_{i_k} \otimes h$$

for vectors  $x = \eta_{i_1} \otimes \dots \otimes \eta_{i_k} \otimes h$ . Define a twisted new Fock space as

$$T_N(H) = \left\{ \sum_{|w| \leq k} w \otimes h_w : w \in \mathbf{F}_N^+, k \geq 1, h_w \in H \right\}$$

as before.

We define a form

$$\langle \cdot, \cdot \rangle_\Phi: T_N(H) \times T_N(H) \longrightarrow \mathbf{C}$$

as follows. For  $w, w' \in \mathbf{F}_N^+$ ,  $h, h' \in H$

- (i)  $\langle e \otimes h, e' \otimes h' \rangle_\Phi = \langle h | h' \rangle$ ;
- (ii) if  $|w| \neq |w'|$  then  $\langle w \otimes h, w' \otimes h' \rangle_\Phi = 0$ ;
- (iii) if  $w = i_1 \dots i_k$ ,  $w' = i'_1 \dots i'_k$ , then

$$\langle w \otimes h, w' \otimes h' \rangle_\Phi = \langle h | \Phi(e_{i_1 \sigma(i'_1)} \otimes \dots \otimes e_{i_k \sigma(i'_k)}) h' \rangle,$$

with  $i(\sigma) = \#\{(i, j) \in \{1, \dots, N\}^2 : i < j, \sigma(i) > \sigma(j)\}$ .

We need the map  $\phi$  being completely positive and then  $\langle \cdot, \cdot \rangle_\Phi$  is positive semi-definite:

$$\tilde{\Phi}(e_i \otimes e_j) = \Phi(e_i \otimes P_q^{(N)} e_j).$$



Let us now turn to the construction of the multiresolution wavelet analysis.

Suppose we have two pairs of filters and then we have two pairs of scaling functions plus wavelet  $\Phi, \Psi$  and  $\hat{\Phi}, \hat{\Psi}$ . They are defined by

$$\begin{aligned}\hat{\Phi}(\xi) &= m_0(\xi/N)\hat{\Phi}(\xi/N) & \hat{\Psi}(\xi) &= m_1(\xi/N)\hat{\Phi}(\xi/N) \\ \hat{\Phi}(\xi) &= \tilde{m}_0(\xi/N)\hat{\Phi}(\xi/N) & \hat{\Psi}(\xi) &= \tilde{m}_1(\xi/N)\hat{\Phi}(\xi/N).\end{aligned}$$

We want to take the direct sum of two MRA. We have the following two sequences of successive approximation spaces  $U_j$  and  $\tilde{U}_j$ . The closed subspaces satisfy

$$\cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots \quad (6)$$

with

$$\begin{aligned}\bigvee_{j \in \mathbf{Z}} V_j &= L^2(\mathbf{C}) & \bigwedge_{j \in \mathbf{Z}} V_j &= \{0\} \\ \cdots \tilde{V}_2 \subset \tilde{V}_1 \subset \tilde{V}_0 \subset \tilde{V}_{-1} \subset \tilde{V}_{-2} \subset \cdots & & & (7)\end{aligned}$$

with

$$\bigvee_{j \in \mathbf{Z}} \tilde{V}_j = L^2(\mathbf{C}) \quad \bigwedge_{j \in \mathbf{Z}} \tilde{V}_j = \{0\}.$$

Formulae (6) and (7) have the additional requirements

$$f \in V_j \Leftrightarrow f(N^j \cdot) \in V_0 \quad \tilde{f} \in \tilde{V}_j \Leftrightarrow \tilde{f}(N^j \cdot) \in \tilde{V}_0$$

i.e., all spaces are scaled versions of the central space  $V_0$  and  $\tilde{V}_0$ , respectively. For every  $j \in \mathbf{Z}$ , define  $W_j$  to be the orthogonal complement of  $V_j$  in  $V_{j-1}$  and  $\tilde{W}_j$  the orthogonal complement of  $\tilde{V}_j$  in  $\tilde{V}_{j-1}$ . We have

$$V_{j-1} = V_j \oplus W_j \quad \tilde{V}_{j-1} = \tilde{V}_j \oplus \tilde{W}_j.$$

Also  $W_j \perp W_{j'}$  if  $j \neq j'$  and  $\tilde{W}_j \perp \tilde{W}_{j'}$  if  $j \neq j'$ . Define a sequence of successive approximation spaces

$$Y_1 = V_1 \oplus \tilde{V}_1 \quad Y_0 = V_0 \oplus \tilde{V}_0 \quad Y_{-1} = V_{-1} \oplus \tilde{V}_{-1} \quad Y_{-2} = V_{-2} \oplus \tilde{V}_{-2}, \dots$$

such that if  $f(t, s)$  is in  $Y_j$  then  $f(Nt, Ns)$  and all  $f(t-k, s-k)$  are in  $Y_{j+1}$ . Define  $\hat{W}_j$  to be the orthogonal complement of  $V_j \cap \tilde{V}_j$  in  $Y_{j-1}$ . Then

$$\hat{W}_j = (V_j \cap \tilde{V}_j)^\perp \cap Y_{j-1} = V_j^\perp \oplus \tilde{V}_j^\perp \cap Y_{j-1}.$$

Thus

$$\begin{aligned}Y_j &= (V_j \oplus \tilde{V}_j) = (V_{j-1} \oplus W_j) \oplus (\tilde{V}_{j-1} \oplus \tilde{W}_j) \\ &= (V_{j-1} \oplus \tilde{V}_{j-1}) \oplus (W_j \oplus \tilde{W}_j) = Y_{j-1} \oplus \hat{W}_j\end{aligned}$$

which implies  $Y_j = Y_{j-1} \oplus \hat{W}_j$ , where we set  $\hat{W}_j = W_j \oplus \tilde{W}_j$ . Then

$$\begin{aligned}\hat{W}_j &= W_j \oplus \tilde{W}_j = (V_j^\perp \cap V_{j-1}) \oplus (\tilde{V}_j^\perp \cap \tilde{V}_{j-1}) \\ &= (V_j^\perp \oplus \tilde{V}_j^\perp) \cap (V_{j-1} \oplus \tilde{V}_{j-1}) = (V_j \cap \tilde{V}_j)^\perp \cap Y_{j-1},\end{aligned}$$

so that  $\hat{W}_j$  is the orthogonal complement of  $V_j \cap \tilde{V}_j$  in  $Y_{j-1}$ .

Thus  $L^2(\mathbf{C}^2) = \bigoplus_{j \in \mathbf{Z}} \hat{W}_j$ . The basic point of MRA is that whenever a collection of closed subspaces satisfies

$$\cdots \subset Y_2 \subset Y_1 \subset Y_0 \subset Y_{-1} \subset Y_{-2} \subset \cdots$$

$$\bigvee_{j \in \mathbf{Z}} Y_j = L^2(\mathbf{C}^2) \quad \bigwedge_{j \in \mathbf{Z}} Y_j = \{0\}$$

$$f \in Y_j \text{ if and only if } f(N^j \cdot) \in Y_0$$

$$f \in Y_0 \text{ then } (\cdot - k) \in Y_0 \text{ for every } k \in \mathbf{Z}$$

there exists  $\phi \in Y_0$  such that  $\{\phi_{0,k} : k \in \mathbf{Z}\}$  is an orthonormal basis in  $Y_0$  where  $\phi_{j,k}(z) = N^{-j} \phi(N^{-j}z - k)$ .

Then there exists an orthonormal wavelet basis  $\{\psi_{j,k} : j, k \in \mathbf{Z}\}$  of  $L^2(\mathbf{C}^2)$ , and we may therefore define the following isomorphism  $\eta: \mathbf{C}^2 \rightarrow \mathcal{H}$ , where  $\mathcal{H}$  denotes the quaternions, by  $\eta(z_1, z_2) = z_1 + z_2 e_2$  and  $e_2 = (0, 0, 1, 0) \in \mathbf{R}^4$ . Consider now the space  $L^2(\mathcal{H})$  equipped with the usual quaternion inner product: given the filters  $m_i$  and  $\tilde{m}_j$  we define the following two matrix functions  $A$  and  $\tilde{A}$  by

$$A_{k,l}(z) = \frac{1}{N} \sum_{w^N=z} w^{-l} m_k(w)$$

and

$$\tilde{A}_{k,l}(z) = \frac{1}{N} \sum_{w^N=z} w^{-l} \tilde{m}_k(w)$$

respectively.

They satisfy the following biorthogonality conditions:

$$\sum_{k=0}^{N-1} \overline{A_{k,i}(z)} \tilde{A}_{k,j}(z) = \delta_{i,j}$$

and

$$\frac{1}{N} \sum_{w^N=z} \overline{m_i(w)} m_j(w) = \delta_{i,j} \quad \frac{1}{N} \sum_{w^N=z} \overline{\tilde{m}_i(w)} \tilde{m}_j(w) = \delta_{i,j}.$$

Take  $B = A \oplus A^*$  and  $\tilde{B} = \tilde{A} \oplus \tilde{A}^*$  the matrix functions associated with the MRA's with filters  $m_i$  and  $\tilde{m}_i$  respectively.

It can be easily checked for  $N = 2$  that the following equation is satisfied:

$$B\tilde{B}^* + \tilde{B}^*B = \mathbf{1}.$$

**Theorem 5.** Let  $S = (S_0, S_1, \dots, S_N)$  and  $\tilde{S} = (\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_N)$  be a pair of wavelet representations on  $H = L^2(\mathbf{C})$  with invertible loop matrices  $A$  and  $\tilde{A}$ , respectively. Let  $\mathbf{S} = (S \quad \tilde{S})$  be the matrix associated with  $S$  and  $\tilde{S}$  and let  $P = \mathbf{S}^* \mathbf{S} + \mathbf{S} \mathbf{S}^*$ .

Let  $T = (T_1, T_2, \dots, T_{N-1}, \tilde{T}_1, \tilde{T}_2, \dots, \tilde{T}_{N-1})$  be the creation operator on  $\mathcal{F}_{\in N}(\mathcal{H}, \mathcal{P})$ . Then:

$$T_i^* T_j|_H = (S_{i-1}^* S_{j-1}) = (AA^*)_{i,j} \quad \tilde{T}_i^* \tilde{T}_j|_H = (\tilde{S}_{i-1}^* \tilde{S}_{j-1}) = (\tilde{A}\tilde{A}^*)_{i,j}$$

$$T_i \tilde{T}_j^*|_H + \tilde{T}_j^* T_i|_H = \delta_{i,j} \mathbf{1} \text{ where } H \text{ denotes the space } L^2(\mathcal{H}).$$

Hence the  $*$ -algebra generated by  $\tilde{T}_j, j = 1, \dots, N$  is an oscillator algebra. It has a representation containing the Cuntz-Toeplitz isometries.

**Proof.** It is easy to check that the fermion algebra relations hold for the direct sum of wavelet representations. □

### 3.2. Construction of $q$ -oscillator algebras

Let us consider the system of wavelet representations  $S = (S_0, S_1, \dots, S_N)$  and  $\tilde{S} = (\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_N)$  depending on a real parameter  $q$  on  $L^2(\mathbf{C})$  with invertible loop matrices  $A$  and  $\tilde{A}$ , respectively.

Let  $S = (S_0, S_1, \dots, S_N)$  and  $\tilde{S} = (\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_N)$  be a pair of wavelet representations on  $L^2(\mathbf{C})$  with invertible loop matrices  $A$  and  $\tilde{A}$ , respectively. Let us assume that one of them, say  $\tilde{S} = (\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_N)$ , depends on a real parameter  $q$  and the matrix  $A$  is given by

$$\tilde{A}_{k,l}(z) = \sum_{w^N=z} (qw)^{-l} \tilde{m}_k(w)$$

where the  $\tilde{m}_j$  are the filters of the MRA depending on a parameter  $q$  as constructed in [7]. Then  $\tilde{A}_{k,l}$  is a unitary matrix function. The unitarity of the  $\{(1 - q^{2N})\tilde{m}_j(tq^j)\}_{i,j=0,\dots,N-1}$  is equivalent to

$$\sum_k \tilde{A}_{i,k}(z) \overline{\tilde{A}_{j,k}(z)} = (1 - q^{2N}) \sum_{w^N=z} \tilde{m}_i(w) \overline{\tilde{m}_j(w)}.$$

We assume then a generalized biorthogonality holds

$$\sum_k A_{i,k}(z) \overline{\tilde{A}_{j,k}(z)} = (1 - q^N) \sum_{w^N=z} m_i(w) \overline{\tilde{m}_j(w)} = \delta_{i,j} 1.$$

Let  $\mathbf{S}$  be the matrix defined as above. Let  $P = \mathbf{S}^* \mathbf{S}$  be the matrix  $2N \times 2N$  in  $B(\mathcal{H})$  determined by the wavelet representations such that

$$\mathbf{S}^* \mathbf{S} = \begin{pmatrix} S^* S & 0 \\ 0 & \tilde{S}^* \tilde{S} \end{pmatrix}$$

Then we have the following theorem.

**Theorem 6.** *Let  $S = (S_0, S_1, \dots, S_N)$  and  $\tilde{S} = (\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_N)$  be a pair of wavelet representations on  $L^2(\mathbf{C})$  with invertible loop matrices  $A$  and  $\tilde{A}$ , respectively. Let us assume that one of them, say  $\tilde{S} = (\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_N)$ , depends on a real parameter  $q$  and the matrix  $A$  is given by*

$$\tilde{A}_{k,l}(z) = \sum_{w^N=z} (qw)^{-l} \tilde{m}_k(w).$$

*Let  $\mathbf{S}$  be as above and let  $P = \mathbf{S}^* \mathbf{S}$  be the matrix  $2N \times 2N$  in  $B(\mathcal{H})$ .*

*Let  $T = (T_1, T_2, \dots, T_{N-1}, \tilde{T}_1, \tilde{T}_2, \dots, \tilde{T}_{N-1})$  be the creation operator in  $\mathcal{F}_{\in N}(\mathcal{H}, \mathcal{P})$ . Then we have:*

$$\begin{aligned} T_i^* T_j|_H &= (S_{i-1}^* S_{j-1}) = (AA^*)_{i,j} & \tilde{T}_i^* \tilde{T}_j|_H &= (\tilde{S}_{i-1}^* \tilde{S}_{j-1}) = (\tilde{A}\tilde{A}^*)_{i,j} \\ \tilde{T}_i T_j^*|_H - T_j^* \tilde{T}_i|_H &= \delta_{i,j} [N]_q \mathbf{1} \end{aligned}$$

**Proof.** As in lemma 3.5 of [6] we have

$$(S_{i-1}^* S_{j-1}) = (AA^*)_{i,j}$$

and similarly

$$(\tilde{S}_{i-1}^* \tilde{S}_{j-1}) = (\tilde{A}\tilde{A}^*)_{i,j}$$

For  $1 \leq i \leq N$ , let  $T_i$  be the  $i$ th creation operators on the twisted Fock space.

To find the action of  $\tilde{T}_i^*$  on the spanning vectors we consider

$$\begin{aligned} \langle \tilde{T}_i(w \otimes h) | \tilde{T}_j(w' \otimes h') \rangle &= \langle iw \otimes h | (jw') \otimes h' \rangle \\ &= \left\langle h \left| \sum_{\sigma} q^{2\sigma(i)} \Phi(e_{i_1, i'_1} \cdots e_{i_k, i'_k}) h' \right. \right\rangle \end{aligned}$$

implying

$$\tilde{T}_j^* \tilde{T}_i = \delta_{i,j} \sum_{\sigma \in \mathcal{S}_N} q^{2\sigma(i)} = \frac{(1 - q^{2N})}{1 - q^2} \delta_{i,j}$$

We consider  $\mathcal{S}_i$  and  $\tilde{\mathcal{S}}_i$  coming from wavelet analysis.

For  $1 \leq i \leq N$  let  $\tilde{T}_i$  and  $\tilde{S}_i$  be the creation operators and the operator coming from the wavelet construction.

Then

$$\tilde{S}_{i-1}^* \tilde{S}_{j-1} = (\tilde{A} \tilde{A}^*)_{i,j} = \tilde{T}_i \tilde{T}_j^*$$

Let us describe the action of  $\tilde{T}_i T_j^*$  on the spanning vectors. Since

$$\langle T_i^*(w \otimes h) \mid w' \otimes h' \rangle = \langle w \otimes h \mid (iw') \otimes h' \rangle$$

Thus  $\langle (\tilde{T}_i T_j^*)(w \otimes h) \mid w' \otimes h' \rangle$  can be computed as follows:

$$\begin{aligned} \langle (\tilde{T}_i T_j^*)(w \otimes h) \mid w' \otimes h' \rangle &= \langle (i, j)^{-1} w \otimes h \mid w' \otimes h' \rangle \\ &\quad \times \langle p_{i,j}^{-1} w_{i_1, i'_1}, w_{i_2, i'_2}, \dots, w_{i_k, i'_k} \otimes h \mid w'_{i_1, i'_1}, w'_{i_2, i'_2}, \dots, w'_{i_k, i'_k} \otimes h' \rangle \\ &= \left\langle \frac{(1 - q^{-N})}{(1 - q^{-1})} (1 - q^{-1}) w_{i_1, i'_1}, w_{i_2, i'_2}, \dots, w_{i_k, i'_k} \otimes h \mid \right. \\ &\quad \left. \times \left[ w'_{i_1, i'_1}, w'_{i_2, i'_2}, \dots, w'_{i_k, i'_k} \otimes h' \right] \right\rangle \end{aligned}$$

On the other side we have

$$\begin{aligned} \langle T_j^* \tilde{T}_i(w \otimes h) \mid w' \otimes h' \rangle &= \langle (iw) \otimes h \mid (jw') \otimes h' \rangle \\ &= \left\langle \frac{(1 - q^N)}{(1 - q)} (1 - q) w_{i_1}, w_{i_2}, \dots, w_{i_k} \otimes h \mid w'_{i'_1}, w'_{i'_2}, \dots, w'_{i'_k} \otimes h' \right\rangle \end{aligned}$$

thus it follows:

$$T_j^* \tilde{T}_i = 1 - q^N$$

and

$$\tilde{T}_i T_j^* = 1 - q^{-N}.$$

Hence

$$\tilde{T}_i T_j^* - T_j^* \tilde{T}_i = \delta_{i,j} [N]_q \mathbf{1}.$$

Then the creation and annihilation operators  $\tilde{T}_i$  and  $T_j^*$  and  $N$ , viewed as a number operator, yield a representation of the  $q$ -oscillator algebra.  $\square$

#### 4. Concluding remarks

We have shown how tools from transform theory and wavelet analysis help us in the construction of new representations of certain  $q$ -relations from statistical mechanics. The  $q$ -relations have been studied earlier in connection with quantum fields [1] and statistical mechanics [12]. In particular, paper [12] serves to show that the  $q$ -relations interpolate between the bosons and the fermions. Further, in [12], the partition function is calculated for the quons and it is established that it exhibits Gibbs' paradox. As a result, the corresponding notions of entropy, free energy and particle number break with our traditional understanding of thermodynamical quantities.

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